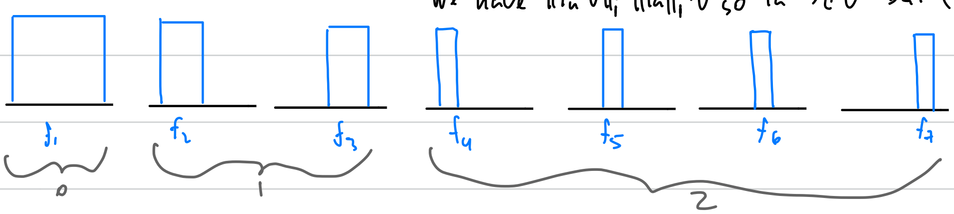
Math 564: Real analysis and measure theory
Lecture 17

Convergence in measure.

Recall the example of L^1-convergence but not pointwise:

$f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/4]}$, $f_5 = \chi_{[1/4,1/2]}$, $f_6 = \chi_{[1/2,3/4]}$, $f_7 = \chi_{[3/4,1]}$, and in general, $f_n = \chi_{[j/2^k,\,(j+1)/2^k]}$ where $n = 2^k + j$ with $0 \le j < 2^k$.

We have $\|f_n - 0\|_1 = \|f_n\|_1 \to 0$ so $f_n \to_{L^1} 0$ but (f_n) doesn't converge pointwise.



However, there are subsequences of (f_n) that converge to 0 a.e., e.g. (f_{2^n}). Turns out this is a general phenomenon: every L^1-convergent sequence admits a subsequence converging a.e. To prove this, we need an intermediate notion of convergence, called **convergence in measure**.

Def. For a measure space (X,μ), $f,g : X \to \overline{\mathbb{R}} := \{\pm\infty\}$ μ-measurable functions, and $\alpha > 0$, put

$$\Delta_\alpha(f,g) := \{x \in X : |f(x) - g(x)| \ge \alpha\}$$
$$\delta_\alpha(f,g) := \mu(\Delta_\alpha(f,g)).$$

Note. For μ-measurable sets $A, B \subseteq X$, $\Delta_\alpha(\mathbb{1}_A, \mathbb{1}_B) = A \Delta B$ and $d_\mu(A,B) := \mu(A \Delta B) = \delta_\alpha(\mathbb{1}_A, \mathbb{1}_B)$, for all $0 < \alpha \le 1$.

The function δ_α doesn't satisfy the Δ-inequality: let $f \equiv 0$, $g \equiv 1$, $h \equiv 2$, then $\delta_2(f,g) = 0 = \delta_2(g,h)$ but $\delta_2(f,h) = \mu(X)$, so δ_α is not a pseudo-metric.

However, the family $\{\delta_\alpha\}_{\alpha>0}$ is "kind of pseudo-metric."

Prop (additive triangle inequality). For all $\alpha, \beta > 0$ and $f, g, h : X \to \overline{\mathbb{R}}$ μ-measurable,

$$\Delta_{\alpha+\beta}(f,h) \subseteq \Delta_\alpha(f,g) \cup \Delta_\beta(g,h),$$
$$\delta_{\alpha+\beta}(f,h) \leq \delta_\alpha(f,g) + \delta_\beta(g,h).$$

Proof. For each $x \in X$,

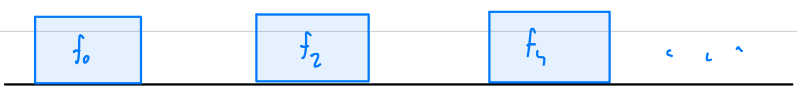
$$|f(x)-h(x)| \leq |f(x)-g(x)| + |g(x)-h(x)|$$

$$\begin{aligned} x \in \Delta_{\alpha+\beta}(f,h) \iff |f(x)-h(x)| \geq \alpha+\beta &\implies |f(x)-g(x)| + |g(x)-h(x)| \geq \alpha+\beta \\ &\implies |f(x)-g(x)| \geq \alpha \text{ or } |g(x)-h(x)| \geq \beta \\ &\implies x \in \Delta_\alpha(f,g) \cup \Delta_\beta(g,h). \end{aligned}$$

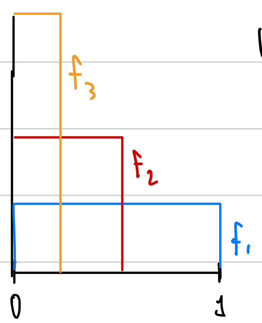
$\square$

Def. For a measure space (X,μ) and μ-measurable functions (f_n) and f, we say that (f_n) **converges in measure** to f, denoted $f_n \to_\mu f$, if $\lim_{n\to\infty} \delta_\alpha(f_n, f) = 0$ for all $\alpha > 0$.

Examples. (a) Let $f_n := \mathbb{1}_{[n,n+1)}$, then $f_n \to 0$ pointwise but not in measure and not in L^1.



(b)



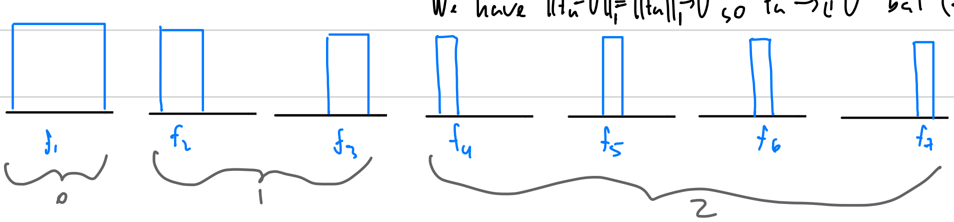
Let $f_n := n^2 \cdot \mathbb{1}_{(0,\frac{1}{n})}$. Then $f_n \to 0$ pointwise but not in L^1 because $\int f_n \, d\lambda = n$.
However, $f_n \to_\lambda 0$ because for each α, $\delta_\alpha(f_n, 0) = \frac{1}{n}$ for all large enough n.

Moral: Convergence in measure doesn't detect how badly f_n differs from the limit, but how large is the place where they differ.

(c) $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/4]}$, $f_5 = \chi_{[1/4,1/2]}$, $f_6 = \chi_{[1/2,3/4]}$, $f_7 = \chi_{[3/4,1]}$, and in general, $f_n = \chi_{[j/2^k,\,(j+1)/2^k]}$ where $n = 2^k + j$ with $0 \leq j < 2^k$.

Also, $f_n \to_\lambda 0$ because $\delta_\alpha(f_n, 0) = 2^{-k} \to 0$, if n is in k^{th} group and $\alpha \leq 1$.

We have $\|f_n - 0\|_1 = \|f_n\|_1 \to 0$ so $f_n \to_{L^1} 0$ but (f_n) doesn't converge pointwise.



The following two facts are the only general implications between these three modes of convergence:

Prop. For any measure space (X,μ), $f_n \to_{L^1} f \implies f_n \to_\mu f$.

Proof. Assume $f_n \to_{L^1} f$ and fix $\alpha > 0$. Then by Chebyschev:
$$\delta_\alpha(f_n, f) = \mu(\{x \in X : |f_n(x) - f(x)| \geq \alpha\}) \leq \frac{1}{\alpha} \cdot \|f - f_n\| \to 0 \quad \text{as } n \to \infty. \quad \square$$

Switch of quantifiers trick: Let (X,μ) be a finite measure space. Let $P_n \subseteq X$ be an increasing sequence of μ-measurable sets. For every $\varepsilon > 0$,
$$\forall x \in X \; \exists n \in \mathbb{N} \; x \in P_n \implies \exists n \in \mathbb{N} \; \forall^\mu_{-\varepsilon} x \in X \quad x \in P_n,$$
where $\forall^\mu_{-\varepsilon} x \in X$ means for all $x \in X \setminus Z$ for some $\mu(Z) \leq \varepsilon$.

Proof. By the left hand side, $\bigcup_{n \in \mathbb{N}} P_n = X$, so $\lim_{n\to\infty} \mu(P_n) = \mu(X)$ hence for a large enough $n \in \mathbb{N}$, $\mu(P_n) \geq \mu(X) - \varepsilon$. $\square$

Prop. In a finite measure space (X,μ), $f_n \to f$ a.e. $\implies f_n \to_\mu f$.

Proof. Discarding a null set, we may assume $f_n \to f$ everywhere. So for each $\alpha > 0$, we have
$$\forall x \in X \; \exists N \in \mathbb{N} \; \forall n \geq N \quad |f_n(x) - f(x)| < \alpha.$$
For any $\varepsilon > 0$, we switch the quantifiers and get:
$$\exists N \in \mathbb{N} \; \forall^\mu_{-\varepsilon} x \in X \; \forall n \geq N \quad |f_n(x) - f(x)| < \alpha.$$
i.e.
$$\exists N \in \mathbb{N} \; \forall n \geq N \; \forall^\mu_{-\varepsilon} x \in X \quad |f_n(x) - f(x)| < \alpha$$
so $\exists N \; \forall n \geq N \quad \delta_\alpha(f_n, f) < \varepsilon$, which means $\lim_{n\to\infty} \delta_\alpha(f_n, f) = 0$.

By a switch of quantifiers trick, one can also prove Egorov's theorem about almost uniform convergence (HW).

Now let's study convergence in measure

Prop (almost uniqueness of limit). In any measure space (X,μ), $f_n \to_\mu f$ and $f_n \to_\mu g$, then $f = g$ a.e.

Proof. For each $\alpha > 0$, we have $\delta_\alpha(f,g) \leq \delta_{\alpha/2}(f, f_n) + \delta_{\alpha/2}(f_n, g) \to 0$ as $n \to \infty$ so $\delta_\alpha(f,g) = 0$ for all $\alpha > 0$. Hence $f = g$ a.e. since $\{x \in X : |f(x) - g(x)| > 0\} = \bigcup_{n \in \mathbb{N}^+} \Delta_{1/n}(f,g)$ and the latter is null. $\square$

Def. Call a sequence (f_n) *Cauchy in measure* if for each $\alpha > 0$,
$$\delta_\alpha(f_n, f_m) \to 0 \quad \text{as} \quad \min(n,m) \to \infty.$$

Prop. (a) If $f_n \to_\mu f$ then (f_n) is Cauchy in measure.

(b) If (f_n) is Cauchy and admits a subsequence $f_{n_k} \to_\mu f$ as $k \to \infty$, then $f_n \to_\mu f$ as $n \to \infty$.

Proof. HW.

Theorem (Completeness of convergence in measure). Every sequence (f_n) which is Cauchy in measure converges in measure, i.e. $\exists$ μ-measurable f s.t. $f_n \to_\mu f$. Moreover, $f_{n_k} \to f$ a.e. for some subsequence (f_{n_k}).

Proof. Note that by part (b) above, we may restrict to any subsequence (accelleration).

Claim 1. WLOG, $\delta_{2^{-n}}(f_n, f_{n+1}) \leq 2^{-n}$ for all n, by restricting to a subsequence.

Pf of Claim. We define (n_k) recursively: let $n_{-1} = 0$, choose $n_k > n_{k-1}$ such that
$$\delta_{2^{-k}}(f_{n_k}, f_n) \leq 2^{-k} \quad \text{for all } n \geq n_k.$$
Such an n_k exists by the Cauchy condition with $\alpha := 2^{-k}$. $\square$ (Claim)

We now show that for a.e. $x \in X$, $(f_n(x))$ is Cauchy.

Claim 2. If $x \notin B_N := \bigcup_{n \geq N} \Delta_{2^{-n}}(f_n, f_{n+1})$ then for all $m \geq n \geq N$,
$$|f_n(x) - f_m(x)| \leq 2^{-(n-1)} \to 0 \text{ as } n \to \infty,$$
so $(f_n(x))$ is Cauchy.

Pf of Claim. $|f_n(x) - f_m(x)| \leq \sum_{i=n}^{m-1} |f_i(x) - f_{i+1}(x)| \leq \sum_{i=n}^{m-1} 2^{-i} \leq \sum_{i=n}^{\infty} 2^{-i} = 2^{-(n-1)}$. □ (Claim)

But $\mu(B_N) \leq \sum_{n \geq N} \mu(\Delta_{2^{-n}}(f_n, f_{n+1})) = \sum_{n \geq N} \delta_{2^{-n}}(f_n, f_{n+1}) \leq \sum_{n \geq N} 2^{-n} = 2^{-(N-1)}$, which is summable, so by Borel-Cantelli, a.e. $x \in X$ is eventually not in B_N, i.e. $\exists N$ such that $x \notin \bigcup_{n \geq N} B_n = B_N$ (the last equality is due to (B_m) being decreasing). Thus by Claim 2, $(f_n(x))$ is Cauchy and let $f(x)$ denote the limit.

The function $f : X \to \overline{\mathbb{R}}$ (defined a.e.) is μ-measurable being a a.e. pointwise limit of μ-measurable functions f_n. It remains to show that $f_n \to_\mu f$. To this end fix $\alpha > 0$ and choose $N \in \mathbb{N}$ so that $2^{-(N-2)} \leq \alpha$. Then $\Delta_\alpha(f_N, f) \subseteq \Delta_{2^{-(N-2)}}(f_N, f)$, and by Claim 2, if $x \notin B_N$ then $|f_N(x) - f(x)| = \lim_{m \to \infty} |f_N(x) - f_m(x)| \leq 2^{-(N-1)} < 2^{-(N-2)}$, so $x \notin \Delta_{2^{-(N-2)}}(f_N, f)$. In other words $\Delta_{2^{-(N-2)}}(f_N, f) \subseteq B_N$, so
$$\delta_\alpha(f_N, f) \leq \delta_{2^{-(N-2)}}(f_N, f) \leq \mu(B_N) \leq 2^{-(N-1)} \to 0 \text{ as } N \to \infty,$$
thus $f_N \to_\mu f$ as $N \to \infty$. □